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The $\Delta I = \frac{3}{2}, \frac{5}{2}$ contributions in $\mathrm{K} \to 2\pi$ decays and $(\delta_0 - \delta_2)$ S-wave π–π phase shifts

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Abstract. The $\Delta I = \frac{3}{2}, \frac{5}{2}$ contributions to the $\mathrm{K} \to 2\pi$ (CP conserving) decay amplitudes due to both the virtual electromagnetic and the intrinsic $\Delta I = \frac{3}{2}$ weak interaction effects are calculated. The electromagnetic contributions are calculated within a model from the pion and kaon electromagnetic mass differences and their total sum can be made finite at the physical point of the amplitudes although they are individually divergent. The $\Delta I = \frac{3}{2}$ weak interaction effect is related to the coefficient of the divergent part of the pion electromagnetic mass difference. The predicted values of the total amplitudes and also that of the S-wave π–π phase shift $(\delta_0 - \delta_2)$ are found to be in good agreement with their experimental values.

1. Introduction

During the last three years the accuracies of the measurements of $\mathrm{K}_\mathrm{S}^0 \to \pi^{\overset{+}{0}} \pi^{\overset{-}{0}}$ and $\mathrm{K}^+ \to \pi^+\pi^0$ decay rates have been considerably improved (Söding *et al* 1972) and it is now possible to discuss the $\Delta I = \frac{1}{2}$ rule, and its deviations in $\mathrm{K} \to 2\pi$ decays, quantitatively. The fact that $\mathrm{K}^+ \to \pi^+\pi^0$ decay exists, albeit small compared to $\mathrm{K}_\mathrm{S}^0 \to 2\pi$ decays, indicates that the $\Delta I = \frac{1}{2}$ rule is violated in $\mathrm{K} \to 2\pi$ decays. That the violation of the $\Delta I = \frac{1}{2}$ rule in $\mathrm{K} \to 2\pi$ decays may be due to the virtual electromagnetic interactions was pointed out by Gell-Mann and Rosenfeld (1957) as early as 1957, although these authors concluded that the electromagnetic effect in $\mathrm{K} \to 2\pi$ decay is unlikely to be large enough to account for the $\mathrm{K}^+ \to \pi^+\pi^0$ decay.

Cabibbo (1964) and Gell-Mann (1964) used the SU(3) symmetry to discuss the $\mathrm{K} \to 2\pi$ decays. In the current–current form of the nonleptonic interaction Hamiltonian, the $\mathrm{K} \to 2\pi$ decay amplitudes could have a $\Delta I = \frac{1}{2}$ violating part from the $\underline{27}$ representation of SU(3). It has now been possible to calculate the $\Delta I = \frac{1}{2}$ violating effects in $\mathrm{K} \to 2\pi$ decays due to both the intrinsic weak interaction parts as well as the virtual electromagnetic interactions within a certain model, although the $\Delta I = \frac{1}{2}$ part of $\mathrm{K} \to 2\pi$ decay itself is least understood.

In SU(3) symmetry that the matrix elements of $\mathrm{K} \to 2\pi$ decays vanish was shown by Cabibbo (1964) and Gell-Mann (1964). In the context of current algebra and soft-pion limits of $\mathrm{K} \to 2\pi$ decays, the analyses were carried out by Suzuki (1966) and Hara and Nambu (1966) who showed that in the specified limit $\mathrm{K} \to 2\pi$ decays obey the $\Delta I = \frac{1}{2}$ rule. However, one observes that for zero-mass pions ($\mathrm{SU(2)} \times \mathrm{SU(2)}$ limit) one must have also the zero-mass kaon because of the four-momentum conservation. It is simple to see that one is then working with the larger $\mathrm{SU(3)} \times \mathrm{SU(3)}$ symmetry and the vanishing

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of all K → 2π matrix elements is then a consequence of the SU(3) result. If however one gives up the four-momentum conservation, then the soft-pion constraints do not determine the K → 2π decay matrix elements uniquely. So from these considerations an understanding of the $\Delta I = \frac{1}{2}$ part of the K → 2π decays is an open problem. We shall however be concerned here only with the $\Delta I = \frac{3}{2}$ and $\frac{5}{2}$ contributions in K → 2π decays.

The electromagnetic interaction Hamiltonian has $\Delta I = 1$ and 2 parts. It will then induce $\Delta I = \frac{3}{2}$ and $\frac{5}{2}$ effects in the K → 2π decays, when combined with the first order weak interactions. To calculate these $\Delta I = \frac{3}{2}$ and $\frac{5}{2}$ effects in K → 2π decays we use a model as shown in the figure 1. The model was first used by Wallace (1969) to calculate

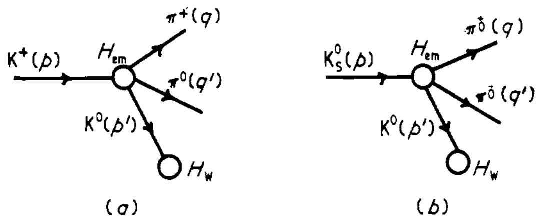

Figure 1. A model for the electromagnetic contributions to K → 2π decays.

the $\Delta I = \frac{3}{2}$ effect in K → 2π decays in the soft-pion limits. We use it to calculate the $\Delta I = \frac{3}{2}$ and $\frac{5}{2}$ contributions to K → 2π decays from the $\Delta I = 2$ electromagnetic mass difference of pions in the soft-kaon limits and the $\Delta I = \frac{3}{2}$ contributions from the $\Delta I = 1$ electromagnetic mass difference of kaons in the soft-pion limits. We then extrapolate the results to the physical points. Although the electromagnetic mass differences of both kaons and pions contain divergent terms, they cancel each other after extrapolation in the contributions to the K → 2π decay amplitudes.

Using certain equal-time commutators of the generalized gauge field model (Lee *et al* 1967) and the Bjorken limits (Bjorken 1966), Müller and Rothlietner (1968) first pointed out the connection between the $\Delta I = \frac{3}{2}$ parts of the K → 2π decay amplitudes due to the weak interactions alone in the soft-kaon limit and the 'divergent' part of the electromagnetic mass difference of pions. To obtain the physical amplitudes one has to extrapolate the result from the soft-kaon point to the physical point (Chen and Mathur 1971).

For the extrapolation we use the technique suggested by Okubo and Mathur (1970). The central idea is that the medium-strong interaction Hamiltonian has a SU(3) × SU(3) symmetry breaking part belonging to the $(\bar{3}, 3)+(3, \bar{3})$ representation, characterized by one parameter 'a', (related to the 'c' parameter of Gell-Mann *et al* (1968) by $a = c/\sqrt{2}$). We now assume that the matrix elements are smooth functions of the parameter 'a' between the values -1 and 2, the former corresponding to the soft-pion limits (SU(2) × SU(2) symmetry) and the latter corresponding to the soft-kaon limit. The 'physical' value corresponds to the value of a ($= -0{\cdot}9$) for which the broken SU(3) × SU(3) description, for instance, leads to the physical masses of the pion and kaon. The point $a = 0$ corresponds to the exact SU(3) × SU(3) symmetry. One should note that the assumption that the matrix elements are smooth functions of the parameter 'a' between $-1 \leqslant a \leqslant 2$ is much stronger than one implicit in the pion and kaon PCAC (partial conservation of the axial-vector current) hypothesis.

It is perhaps worthwhile to mention here the various other methods to calculate the $\Delta I = \frac{3}{2}, \frac{5}{2}$ effects in $K \rightarrow 2\pi$ decays and comment on them. Based on an intrinsic $\Delta I = \frac{3}{2}$ part in weak interactions the $K^+ \rightarrow \pi^+\pi^0$ decay has been discussed by Schwinger (1964). Nieh (1968) used a phenomenological $\Delta I = \frac{3}{2}$ Lagrangian, while Sakurai (1967) used K^* pole model to correlate the various observed $\Delta I = \frac{3}{2}$ effects in nonleptonic decays. Okubo *et al* (1967) used the algebra of currents to calculate the subtraction constant in a once-subtracted dispersion relation for the $K \rightarrow 2\pi$ decay amplitudes. These authors obtained the result that the amplitude for $K^+ \rightarrow \pi^+\pi^0$ is proportional to the pion electromagnetic mass difference Δm_π^2, although they have not taken into account the electromagnetic effects explicitly in their calculation, a comment first made by Feynman (1968). One has also the old η-pole model of Riazuddin and Fayyazuddin (1962), further elaborated by Fäldt *et al* (1967) to calculate the $\Delta I = \frac{3}{2}$ effects in $K \rightarrow 2\pi$ decays due to the virtual electromagnetic interactions. This model by itself does not have $\Delta I = \frac{5}{2}$ effects and, further, one has to use the SU(3) symmetry argument to relate the amplitude for $K \rightarrow \pi\eta$ to that of $K \rightarrow 2\pi$. But we have already mentioned that from the SU(3) symmetry arguments the amplitudes for $K \rightarrow 2\pi$ decays vanish. Greenberg (1969) used involved pole models to calculate the $\Delta I = \frac{3}{2}$ effects in $K \rightarrow 2\pi$ decays, although many parameters in the model are not very well known.

The plan of the paper is as follows. In § 2 we analyse the recent $K \rightarrow 2\pi$ decay rates in terms of the most general parametrization. In §§ 3 and 4 we calculate the $\Delta I = \frac{3}{2}, \frac{5}{2}$ effects in $K \rightarrow 2\pi$ decay amplitudes from the virtual electromagnetic and intrinsic weak interactions. The extrapolations of the matrix elements to the 'physical point' are done in § 5. We give the final results in § 6 and discuss them.

2. Analysis of K → 2π decays

The Hamiltonian responsible for the decay $K \rightarrow 2\pi$ (CP conserving parts only) has the following components (considering only the isospin violating properties):

$$H_W^{NL} = H^{(\Delta I = 1/2)} + H^{(3/2)} + H^{(5/2)}. \tag{2.1}$$

We denote the relevant $K_S^0 \rightarrow \pi^+\pi^-$, $K_S^0 \rightarrow 2\pi^0$ and $K^+ \rightarrow \pi^+\pi^0$ decay amplitudes by A^0_{+-}, A^0_{00} and A^+_{+0} respectively. In terms of the reduced matrix elements

$$a_0 = \langle I = 0 \| H^{(1/2)} \| \tfrac{1}{2} \rangle \tag{2.2}$$

$$a_m = \langle I = 2 \| H^{(m/2)} \| \tfrac{1}{2} \rangle, \qquad m = 3, 5 \tag{2.3}$$

and the S-wave π–π phase shifts, $\delta_I(I = 0, 2)$, the amplitudes A^0_{+-}, A^0_{00} and A^+_{+0} can be expressed as (assuming CTP and CP invariances)

$$A^0_{+-} = 2a_0\, e^{i\delta_0} + (a_3 + a_5)\, e^{i\delta_2} \tag{2.4}$$

$$A^0_{00} = 2a_0\, e^{i\delta_0} - 2(a_3 + a_5)\, e^{i\delta_2} \tag{2.5}$$

$$A^+_{+0} = (\tfrac{3}{2}a_3 - a_5)\, e^{i\delta_2}, \tag{2.6}$$

using appropriate Clebsch–Gordon coefficients. We introduce the parameters

$$\xi = \frac{\frac{3}{2}a_3 - a_5}{a_3 + a_5}, \qquad \omega = \frac{a_3 + a_5}{2a_0} \exp\{i(\delta_2 - \delta_0)\}, \tag{2.7}$$

and the ratios of the squares of the amplitudes A^0_{+-}, A^0_{00} and A^+_{+0}

$$B(K^0_S) = \frac{|A^0_{+-}|^2}{|A^0_{00}|^2} - 1, \tag{2.8}$$

$$b^+ = \frac{|A^+_{+0}|^2}{|A^0_{+-}|^2 + \frac{1}{2}|A^0_{00}|^2}. \tag{2.9}$$

Noting that the reduced matrix elements a_i ($i = 0, 3, 5$) are real from CP invariance, the ratios $B(K^0_S)$ and b^+ can be expressed in terms of the three parameters $|\xi|$, $|\omega|$ and Re ω of (2.7) as

$$B(K^0_S) = \frac{6\,\mathrm{Re}\,\omega - 3|\omega|^2}{1 - 4\,\mathrm{Re}\,\omega + 4|\omega|^2} \tag{2.10}$$

$$b^+ = \frac{2}{3}|\xi|^2 \frac{|\omega|^2}{1 + 2|\omega|^2}. \tag{2.11}$$

From experiments (Söding *et al* 1972) we have

$$\frac{\Gamma(K^0_S \to \pi^+\pi^-)}{\Gamma(K^0_S \to \pi^0\pi^0)} = 2{\cdot}213 \pm 0{\cdot}041 \tag{2.12}$$

$$\frac{\Gamma(K^+ \to \pi^+\pi^0)}{\Gamma(K^0_S \to \pi^+\pi^-) + \Gamma(K^0_S \to \pi^0\pi^0)} = (1{\cdot}458 \pm 0{\cdot}030) \times 10^{-3}. \tag{2.13}$$

Taking out the appropriate phase-space factors from the decay rates $\Gamma(K^0_S \to \pi^+\pi^-)$, $\Gamma(K^0_S \to \pi^0\pi^0)$ and $\Gamma(K^+ \to \pi^+\pi^0)$ we have for the ratios $B(K^0_S)$ and b^+ (Sarker 1972),

$$B(K^0_S)_{\mathrm{exp}} = 0{\cdot}1234 \pm 0{\cdot}0160 \tag{2.14}$$

$$b^+_{\mathrm{exp}} = (1{\cdot}447 \pm 0{\cdot}033) \times 10^{-3}. \tag{2.15}$$

We note that from the $\Delta I = \frac{1}{2}$ rule follows:

$$B(K^0_S) = 0, \qquad \cdot b^+ = 0; \tag{2.16}$$

and if we assume $\Delta I = \frac{5}{2}$ to be absent, then ξ is $\frac{3}{2}$.

The values of the parameter $|\omega|$ and the π–π S-wave phase shift $(\delta_0 - \delta_2)$, depending on the values of $|\xi|$, obtained from (2.10), (2.11), (2.14) and (2.15), are shown in figure 2. In figure 2(*b*) the broken curves indicate the errors of the predicted values of $|(\delta_0 - \delta_2)|$ and the shaded region is the overlap of the predicted values of $|(\delta_0 - \delta_2)|$ and $(\delta_0 - \delta_2) = 55{\cdot}5° \pm 5{\cdot}1°$ at $\sqrt{s} = m_K$, obtained from the analyses of the total and differential cross sections of the π–π scattering cross sections (Marakov *et al* 1972). We note that $(\delta_0 - \delta_2) = 55{\cdot}5° \pm 5{\cdot}1°$ corresponds to the values of $|\xi| = 1{\cdot}35^{+0{\cdot}44}_{-0{\cdot}37}$, indicating that $\Delta I = \frac{5}{2}$ contributions to K → 2π decays are small.

3. Electromagnetic contributions to K → 2π decays

To calculate the virtual electromagnetic contributions to K → 2π decays we use the models of figure 1. The electromagnetic interaction Hamiltonian has $\Delta I = 1$ and 2 parts. When combined with the first-order weak interactions, it will induce $\Delta I = \frac{3}{2}$ and $\frac{5}{2}$ effects in the K → 2π decays.

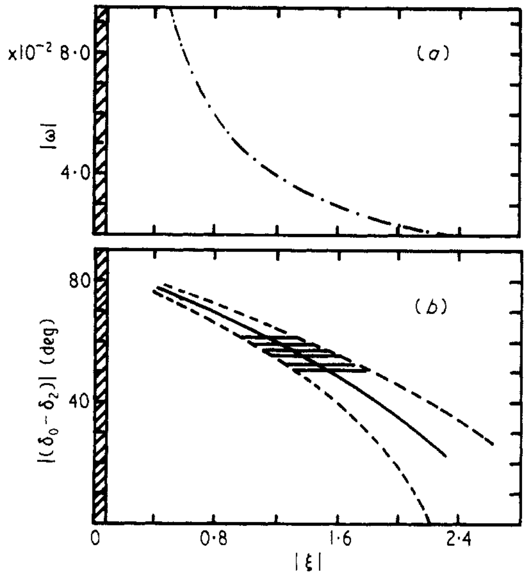

Figure 2. Graphs for (*a*) $|\omega|$ and (*b*) $|(\delta_0-\delta_2)|$ against $|\xi|$. The broken curves in (*b*) indicate the errors. The experimental value $(\delta_0-\delta_2) = 55{\cdot}5^\circ \pm 5{\cdot}1^\circ$ corresponds to $|\xi| = 1{\cdot}35^{+0\cdot44}_{-0\cdot37}$.

To second order in perturbation theory the matrix elements of the electromagnetic interaction Hamiltonian of the graphs of figure 1 are given by

$$M_{\mathrm{el}}(\mathrm{K}^+ \to \mathrm{K}^0\pi^+\pi^0)$$
$$= \frac{\mathrm{i}^2e^2}{2!}(2\pi)^6(16p_0p'_0q_0q'_0)^{1/2}\int \mathrm{d}^4x\, d_{\mu\nu}(x)$$
$$\times \langle \mathrm{K}^0(p')\pi^+(q)\pi^0(q')|T\{V_\mu^{\mathrm{el}}(x)V_\nu^{\mathrm{el}}(0)\}|\mathrm{K}^+(p)\rangle \tag{3.1}$$

$$M_{\mathrm{el}}(\mathrm{K}_\mathrm{S}^0 \to \mathrm{K}^0\pi\pi)$$
$$= \frac{\mathrm{i}^2e^2}{2!}(2\pi)^6(16p_0p'_0q_0q'_0)^{1/2}\int \mathrm{d}^4x\, d_{\mu\nu}(x)$$
$$\times \langle \mathrm{K}^0(p')\pi(q)\pi(q')|T\{V_\mu^{\mathrm{el}}(x)V_\nu^{\mathrm{el}}(0)\}|\mathrm{K}_\mathrm{S}^0(p)\rangle \tag{3.2}$$

where $d_{\mu\nu}(x)$ is the photon propagator. In (3.1) and (3.2) we take soft-kaon approximations to pick up the $\Delta I = 2$ electromagnetic contributions from the pion electromagnetic mass difference, while the soft-pion approximations gives the $\Delta I = 1$ electromagnetic contributions from the kaon electromagnetic mass difference. The extrapolation of these results to the physical points will be considered in § 5.

3.1. $\Delta I = 2$ contributions

We perform a soft-kaon reduction of the two kaons in (3.1) and averaging over the two orders of partial integration obtain, after using the PCAC relation $\partial_\mu A_\mu^{4+\mathrm{i}5} = \sqrt{2}f_\mathrm{K}m_\mathrm{K}^2\mathrm{K}^{4+\mathrm{i}5}$, etc and the appropriate equal-time current commutation relations

$$\lim_{p,p'\to 0} M_{\mathrm{el}}^{(2)}(K^+ \to K^0\pi^+\pi^0)$$
$$= \frac{+e^2}{8f_K^2}(2\pi)^3(2q_0)\int d^4x\, d_{\mu\nu}(x) \times \langle \pi^+(q)\pi^0(q)|T\{V_\mu^{1-i2}(x)V_\nu^3(0)+V_\mu^3(x)V_\nu^{1-i2}(0)\}|0\rangle. \tag{3.3}$$

After performing an isotopic spin rotation on the right-hand side of (3.3) and then assuming crossing symmetry and analyticity we obtain

$$\lim_{p,p'\to 0} M_{\mathrm{el}}^{(2)}(K^+ \to K^0\pi^+\pi^0)$$
$$= \frac{-e^2}{4\sqrt{2}f_K^2}(2\pi)^4(2q_0)\int d^4x\, d_{\mu\nu}(x)[\langle \pi^+(q)|T\{V_\mu^3(x)V_\nu^3(0)\}|\pi^+(q)\rangle - (\pi^+ \to \pi^0)]. \tag{3.4}$$

We have

$$\Delta m_\pi^2 \equiv -\frac{ie^2}{2!}(2\pi)^3(2q_0)\int d^4x\, d_{\mu\nu}(x)[\langle \pi^+(q)|T\{V_\mu^3(x)V_\nu^3(0)\}|\pi^+(q)\rangle - (\pi^+ \to \pi^0)]. \tag{3.5}$$

Using (3.5) in (3.4) we get

$$\lim_{p,p'\to 0} M_{\mathrm{el}}^{(2)}(K^+ \to K^0\pi^+\pi^0) = -\frac{i}{2\sqrt{2}f_K^2}\Delta m_\pi^2. \tag{3.6}$$

Similarly

$$\lim_{p,p'\to 0} M_{\mathrm{el}}^{(2)}(K_S^0 \to K^0\pi\pi) = 0. \tag{3.7}$$

We note that the process of taking the soft-kaon approximation of $M_{\mathrm{el}}(K^+ \to K^0\pi^+\pi^0)$ picks out the $\Delta I = 2$ electromagnetic contributions on the right-hand side of (3.4).

3.2. The $\Delta I = 1$ electromagnetic contributions

We now reduce the two pions on the right-hand side of (3.1) and averaging over the two orders of partial integrations obtain, after using the pion PCAC relation, $\partial_\mu A_\mu^3 = f_\pi m_\pi^2 \pi^3$, etc and the appropriate equal-time current commutation relations

$$\lim_{q,q'\to 0} M_{\mathrm{el}}^{(1)}(K^+ \to K^0\pi^+\pi^0)$$
$$= -\frac{e^2}{4\sqrt{6}f_\pi^2}(2\pi)^3(2p_0)\int d^4x d_{\mu\nu}(x) \times \langle K^0(p')|T\{V_\mu^{1-i2}(x)V_\nu^8(0)+V_\mu^8(x)V^{1-i2}\}|K^+(p)\rangle. \tag{3.8}$$

Again performing an isotopic spin rotation on the right-hand side of (3.8) and using the kaon electromagnetic mass difference relation

$$\Delta m_K^2 = -\frac{ie^2}{2!\sqrt{3}}(2\pi)^3(2p_0)\int d^4x\, d_{\mu\nu}(x)\,[\langle K^+(p)|T\{V_\mu^3(x)V_\nu^8(0)\}|K^+(p)\rangle - (K^+ \to K^0)] \tag{3.9}$$

in (3.8) we obtain

$$\lim_{q,q'\to 0} M_{\mathrm{el}}^{(1)}(\mathrm{K}^+ \to \mathrm{K}^0\pi^+\pi^0) = -\frac{\mathrm{i}}{2\sqrt{2}f_\pi^2}\Delta m_{\mathrm{K}}^2. \tag{3.10}$$

We just point out that Δm_{K}^2 contains only the $\Delta I = 1$ electromagnetic effects.

Later on we will have to extrapolate the results of equations (3.6) and (3.10) from the indicated limiting points to the physical points. In order to do that we need explicit analytic expressions for the electromagnetic mass differences Δm_π^2 and Δm_{K}^2. From the algebra of currents and the hard-pion technique of Schnitzer and Weinberg (1967), Gerstein *et al* (1967) derived the following expression for the pion electromagnetic mass difference:

$$\Delta m_\pi^2 = C_\pi + \frac{3\alpha}{4\pi}m_\pi^2\left(\frac{1+\delta^2}{8}\ln\Lambda^2 + \tfrac{19}{2}\ln 2 - \tfrac{5}{2} + \ln\frac{m_\rho^2}{m_\pi^2} + 5\delta(\tfrac{1}{2}\ln 2 - \tfrac{1}{4}) + \delta^2(-\tfrac{3}{4}+\tfrac{1}{4}\ln 2)\right) \tag{3.11}$$

where Λ is a cut-off parameter of infinite integration and

$$C_\pi = \frac{3\alpha}{4\pi}m_\rho^2 2\ln 2 \tag{3.12}$$

and δ is a parameter of the π–ρ–A_1 system with values $0 \leqslant \delta^2 \leqslant \frac{1}{4}$. We note that in the limit $m_\pi^2 \to 0$, the expression (3.11) is finite and gives $\Delta m_\pi \simeq 5$ MeV compared to the $(\Delta m_\pi)_{\mathrm{exp}} = 4{\cdot}61$ MeV.

The kaon electromagnetic mass difference is however somewhat problematic because of the fact that it is difficult to explain its experimental value as being due to only the conventional electromagnetic interactions (apart from the other difficulty of infinity). So one has to take into consideration some tadpole-type of electromagnetic interactions (Coleman and Glashow 1964). To calculate the actual value of this tadpole-type of contribution to the kaon electromagnetic mass difference is quite model dependent. However for our present purpose we take the view that the difference between the experimental pion and kaon electromagnetic mass differences is given by this tadpole contribution to Δm_{K}^2 (Wallace 1970):

$$(\Delta m_{\mathrm{K}}^2)_{\mathrm{t}} = (m_{\mathrm{K}^+}^2 - m_{\mathrm{K}^0}^2) - (m_{\pi^+}^2 - m_{\pi^0}^2). \tag{3.13}$$

For the non-tadpole contributions to Δm_{K}^2 we take

$$(\Delta m_{\mathrm{K}}^2)_{\mathrm{nt}} = C_{\mathrm{K}} + \frac{3\alpha}{4\pi}m_{\mathrm{K}}^2\left(\frac{1+\delta^2}{8}\ln\Lambda^2 + \tfrac{19}{2}\ln 2 + \ln\frac{m_\rho^2}{m_{\mathrm{K}}^2} + 5\delta(\tfrac{1}{2}\ln 2 - \tfrac{1}{4}) + \delta^2(-\tfrac{3}{4}+\tfrac{1}{4}\ln 2)\right) \tag{3.14}$$

where C_{K} is given by (Dashen 1969)

$$C_{\mathrm{K}} = C_\pi. \tag{3.15}$$

We shall consider the relation (3.15) to be true only in the numerical sense. Dashen showed that the relation (3.15) is also true for the symmetry-breaking Hamiltonian (5.1), considered later on in this work.

4. The $\Delta I = \frac{3}{2}$ weak-interaction effects

We take the nonleptonic weak-interaction Hamiltonian in the Cabibbo form as

$$H_{\mathrm{W}}^{(3/2)}(x) = \frac{G}{\sqrt{2}}\frac{1}{2}(J_{\mu}^{1+i2}(x)J_{\mu}^{4-i5}(x)+J_{\mu}^{4-i5}(x)J_{\nu}^{1+i2}(x))\sin\theta\cos\theta \tag{4.1}$$

where $J_{\mu}(x) = (V(x)+A(x))_{\mu}$ and θ is the Cabibbo angle. We note that only the parity-violating terms in (4.1) contribute to the $\mathrm{K}^{+} \to \pi^{+}\pi^{-}$ decay amplitude M_{+0}^{+} defined by

$$M_{+0}^{+} = -\mathrm{i}(2\pi)^{9/2}(8p_0q_0q_0')^{1/2}\langle\pi^{+}(q)\pi^{0}(q')|H_{\mathrm{W}}^{(3/2)}(0)|\mathrm{K}^{+}(p)\rangle. \tag{4.2}$$

We reduce the kaon on the right-hand side of (4.2) and using the kaon PCAC and the equal-time current commutation relations obtain in the soft-kaon limit:

$$\lim_{p\to 0} M_{+0}^{+} = -\frac{1}{4f_{\mathrm{K}}}(2\pi)^3(2q_0)\langle\pi^{+}(q)\pi^{0}(q)|J_{\mu}^{1+i2}(0)J_{\mu}^{3}(0)+J_{\mu}^{3}(0)J_{\mu}^{1+i2}(0)|0\rangle. \tag{4.3}$$

Making an isotopic spin rotation on the right-hand side of (4.3) and using crossing symmetry and analyticity we obtain from (4.3)

$$\lim_{p\to 0} M_{+0}^{+} = \frac{G\sin\theta\cos\theta}{2\sqrt{2}f_{\mathrm{K}}}(2\pi)^3(2q_0)[\langle\pi^{+}(q)|J_{\mu}^{3}(0)J_{\mu}^{3}(0)|\pi^{+}(q)\rangle-(\pi^{+}\to\pi^{0})]. \tag{4.4}$$

We note that only the parity-conserving parts of $J_{\mu}J_{\mu}$ contribute on the right-hand sides of (4.3) and (4.4). We define

$$(2\pi)^3(2q_0)[\langle\pi^{+}(q)|J_{\mu}^{3}(0)J_{\nu}^{3}(0)|\pi^{+}(q)\rangle-(\pi^{+}\to\pi^{0})] = \delta_{\mu\nu}A+\frac{q_{\mu}q_{\nu}}{m_{\pi}^{2}}B \tag{4.5}$$

where A and B are two constants. From (4.4) and (4.5) we get

$$\lim_{p\to 0} M_{+0}^{+} = \frac{G}{2\sqrt{2}f_{\mathrm{K}}}\sin\theta\cos\theta(4A-B). \tag{4.6}$$

The constants A and B are related to the coefficient of the divergent term of the pion electromagnetic mass difference, first shown by Müller and Rothleither (1968). So to obtain $4A-B$ we proceed as follows. To order e^2 the pion electromagnetic difference (3.5) in momentum space is

$$\Delta m_{\pi}^{2} = -\frac{e^2}{2!}(2\pi)^3(2q_0)\frac{1}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{k^2-\mathrm{i}\epsilon}T_{\mu\mu}(q,k) \tag{4.7}$$

where

$$T_{\mu\nu}(q,k) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}kx}[\langle\pi^{+}(q)|T\{V_{\mu}^{\mathrm{el}}(x)V_{\nu}^{\mathrm{el}}(0)\}|\pi^{+}(q)\rangle-(\pi^{+}\to\pi^{0})]. \tag{4.8}$$

From covariance $T_{\mu\nu}(q,k)$ can be expressed as

$$T_{\mu\nu}(q,k) = (k_{\mu}k_{\nu}+k^2\delta_{\mu\nu})T_1(k^2,\nu)+[(k\,.\,q)^2\delta_{\mu\nu}+k^2q_{\mu}q_{\nu}-(k\,.\,q)(k_{\mu}q_{\nu}+q_{\mu}k_{\nu})]T_2(k^2,\nu)$$

where $\nu = -(k\,.\,q)/m_\pi$. The Bjorken limit (Bjorken 1966) of $T_{\mu\nu}(q,k)$ is given by

$$\lim_{\substack{k_0\to\infty\\ \boldsymbol{k}=0}} T_{\mu\nu} = \mathrm{i}\int \mathrm{d}^3x\mathrm{i}\Bigg(\frac{1}{k_0}\Big\langle \pi^+(q)|[V_\mu^{\mathrm{el}}(\boldsymbol{x},0), V_\nu^{\mathrm{el}}(0)] - \frac{\mathrm{i}}{k_0^2}\left[\frac{\partial V_\mu^{\mathrm{el}}(x,t)}{\partial t}\bigg|_{t=0}, V_\nu^{\mathrm{el}}(0)\right]$$

$$+\mathrm{O}\left(\frac{\mathrm{i}}{k_0^3}\right)\bigg|\pi^+(q)\Big\rangle - (\pi^+\to\pi^0)\Bigg). \tag{4.10}$$

In the generalized gauge field model (Lee *et al* 1967) the first commutator on the right-hand side of (4.10) is zero, while the second commutator is given by

$$\left[\frac{\partial V_\mu^{\mathrm{el}}(\boldsymbol{x},t)}{\partial t}\bigg|_{t=0}, V_\nu^{\mathrm{el}}(0)\right]$$

$$= -\frac{\mathrm{i}}{2f_\pi^2}\delta(\boldsymbol{x})\delta_{\mu k}\delta_{\nu l}\sum_{\alpha=1,2,4,5}(V_k^\alpha(0)V^\alpha(0)+A_k^\alpha(0)A_l^\alpha(0)), \qquad i,j,k,\text{ etc} = 1,2,3. \tag{4.11}$$

Now the right-hand side of (4.11) can be rewritten as

$$\sum_{\alpha=1,2,4,5}(V_k^\alpha V_l^\alpha + A_k^\alpha A_l^\alpha) = +\sum_{\alpha=1,2,3,4,5,6,7,8}(V_k^\alpha V_l^\alpha + A_k^\alpha A_l^\alpha) - \sum_{\alpha=3,6,7,8}(V_k^\alpha V_l^\alpha + A_k^\alpha A_l^\alpha). \tag{4.12}$$

Now the first sum and also the terms $\alpha = 6, 7, 8$ do not contribute to the π–π matrix element on the right-hand side of (4.10) and we obtain from (4.5), (4.10), (4.11) and (4.12)

$$\lim_{\substack{k_0\to\infty\\ \boldsymbol{k}=0}} T_{\mu\nu} = \frac{\mathrm{i}}{k_0^2}\frac{1}{2f_\pi^2}\frac{1}{(2q_0)(2\pi)^3}\delta_{\mu j}\delta_{\nu l}\left(\delta_{jl}A + \frac{q_j q_l}{m_\pi^2}B\right) + \mathrm{O}\left(\frac{1}{q^4}\right). \tag{4.13}$$

To express $T_1(\nu,k^2)$ and $T_2(\nu,k^2)$ in terms of the constants A and B we use the following identifications from the nonrelativistic frame to the covariant reference frame:

$$\mathrm{i}\frac{\delta_{\mu 4}}{k_0}\to -\frac{k_\mu}{k^2}, \qquad \delta_{ij}\to\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right) \tag{4.14}$$

$$k_0^2\frac{q_i q_j}{m_\pi^2}\to -\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{(k\,.\,q)^2}{m_\pi^2}+\frac{1}{m_\pi^2}\{(k\,.\,q)^2\delta_{\mu\nu}+k^2q_\mu q_\nu-(k\,.\,q)(k_\mu q_\nu+q_\mu k_\nu)\}\,. \tag{4.15}$$

Using (4.14) and (4.15) in (4.13) and comparing the resultant expression with (4.9) we obtain

$$\lim_{\substack{k_0\to\infty\\ \boldsymbol{k}=0}} k^2T_1 = \frac{\mathrm{i}}{2f_\pi^2}\frac{1}{(2q_0)(2\pi)^3}\left(-\frac{A}{k^2}+\frac{\nu^2B}{k^4}\right) \tag{4.16}$$

$$\lim_{\substack{k_0\to\infty\\ \boldsymbol{k}=0}} k^2T_2 = \frac{\mathrm{i}}{2f_\pi^2}\frac{1}{(2q_0)(2\pi)^3}\frac{B}{k^2m_\pi^2} \tag{4.17}$$

$$\lim_{\substack{k_0\to\infty\\ \boldsymbol{k}=0}} T_{\mu\mu} = \frac{\mathrm{i}}{2f_\pi^2}\frac{1}{(2q_0)(2\pi)^3}\left(\frac{3A-B}{k^2}-\frac{\nu^2B}{k^4}\right). \tag{4.18}$$

Substituting (4.18) in (4.7) and performing the d^4k integration we obtain

$$(\Delta m_\pi^2)_{\mathrm{div}} = \frac{3\alpha}{32\pi}\frac{1}{2f_\pi^2}(4A-B)\ln\Lambda^2 \tag{4.19}$$

where Λ is the cut-off limit of the infinite integration. Comparing (4.19) with (3.11) we get

$$4A-B = 2f_\pi^2 m_\pi^2(1+\delta^2) \tag{4.20}$$

and using (4.20) in (4.6) we have

$$\lim_{p\to 0} M_{+0}^{+} = \frac{G}{\sqrt{2}}\frac{f_\pi^2}{f_{\mathrm{K}}} m_\pi^2(1+\delta^2). \tag{4.21}$$

5. SU(3) × SU(3) symmetry breaking and mass extrapolation

Following Gell-Mann, Renner and Oakes, we assume that the medium-strong interaction Hamiltonian H_{MS} has two parts; one part H_0, which is invariant under the SU(3) × SU(3) symmetry group, and the other H' that breaks symmetry and belongs to the $(\bar{3}, 3)+(3, \bar{3})$ representation of SU(3) × SU(3). So the most general form for H_{MS} is

$$H_{\mathrm{MS}} = H_0 + u_0 + \sqrt{2}au_8 \tag{5.1}$$

where u_i are the scalar densities belonging to $(\bar{3}, 3)+(3, \bar{3})$, and '$a$' is a parameter. We note that at least in some chiral lagrangian models the symmetry breaking Hamiltonian, as taken in (5.1), leaves the masses of the vector and axial-vector mesons separately, as well as the equal-time commutation relation of the generalized gauge field model as given in (4.11), unchanged (Gottlieb *et al* 1972). The rest masses of the pseudoscalar mesons P_i are however given by

$$m_i^2 = \int \mathrm{d}^3x \langle P_i(k)|H_{\mathrm{MS}}(x)|P_i(k)\rangle. \tag{5.2}$$

One has the general form

$$\langle P_i(k)|u_j|P_l(k)\rangle = \alpha\delta_{j0}\delta_{il} + \beta d_{ijl} \tag{5.3}$$

where $i, l = 1, \ldots 8$, $j = 0, \ldots 8$, and α and β are two parameters related to the pseudoscalar masses. From (5.1), (5.2) and (5.3) we get

$$m_\pi^2 = m_0^2(1+a) \tag{5.4}$$

$$m_{\mathrm{K}}^2 = m_0^2(1-\tfrac{1}{2}a) \tag{5.5}$$

where m_0^2 is some common mass. We first note that in the limit $a \to -1$, $m_\pi^2 = 0$ and the exact SU(2) × SU(2) symmetry is realized. In the limit $a \to 2$, however, $m_{\mathrm{K}}^2 = 0$. The physical masses of m_π^2 and m_{K}^2 determine the value of the parameter 'a' to be $-0{\cdot}9$. We further assume that the pseudoscalar meson masses and the matrix elements depending explicitly on these masses are smooth functions of the parameter 'a' between the limits $a = -1, 2$. For $a = 0$, one of course realizes the full SU(3) × SU(3) symmetry. In the following we would also use the result

$$f_\pi = f_{\mathrm{K}} \tag{5.6}$$

which is consistent with the symmetry breaking assumption (5.1).

We note from (3.6) and (3.10) that the matrix elements $\lim_{p,p'\to 0} M_{\mathrm{el}}^{(2)}(\mathrm{K}^+ \to \mathrm{K}^0\pi^+\pi^0)$ and $\lim_{q,q'\to 0} M_{\mathrm{el}}^{(1)}(\mathrm{K}^+ \to \mathrm{K}^0\pi^+\pi^0)$ are related to the electromagnetic mass differences Δm_π^2 and Δm_{K}^2 in the limits $a \to 2$ and $a \to -1$ respectively. Similarly M_{+0}^{+} in (4.21) is related to m_π^2 in the limit $a \to 2$. We consider the extrapolation of the relation (4.21) first. We have

$$\lim_{a\to 2} M^+_{+0} = \frac{G}{\sqrt{2}} f_\pi (1+\delta^2) m_\pi^2 (a=2) \tag{5.7}$$

$$\lim_{a\to 0} M^+_{+0} = 0. \tag{5.8}$$

From (5.4), (5.5), (5.7) and (5.8) we get for M^+_{+0} at the physical point $a = -0{\cdot}9$

$$M^+_{+0} = \frac{G}{\sqrt{2}} f_\pi (1+\delta^2)(m_\pi^2 - m_K^2). \tag{5.9}$$

This result was first obtained by Chen and Mathur (1970). From the relations (3.6) and (3.10) we have

$$\lim_{a=2} M_{el}^{(2)}(K^+ \to K^0\pi^+\pi^0) = -\frac{i}{2\sqrt{2}f_K^2}\Delta m_\pi^2 (a=2) \tag{5.10}$$

$$\lim_{a=-1} M_{el}^{(1)}(K^+ \to K^0\pi^+\pi^0) = -\frac{i}{2\sqrt{2}f_\pi^2}\Delta m_K^2 (a=-1). \tag{5.11}$$

Without any further knowledge of the 'a' dependence of $M_{el}^{(2)}$ and $M_{el}^{(1)}$, the extrapolations of (5.10) and (5.11) are not uniquely determined and the infinities of the electromagnetic mass differences (3.11) and (3.14) cannot be related. However for (5.10) and (5.11) we take the same extrapolation as that of M^+_{+0}, which makes $M_{el}^{(2)}$ and $M_{el}^{(1)}$ finite at $a = 0$. At the physical point $a = -0{\cdot}9$ we have for $M_{el}^{(2)}$ and $M_{el}^{(1)}$

$$M_{el}^{(2)}(K^+ \to K^0\pi^+\pi^0) = -\frac{i}{2\sqrt{2}f_\pi^2} C_\pi + (m_\pi^2 - m_K^2)\left(\frac{1+\delta^2}{8}\ln\Lambda^2 + d_\pi\right) \tag{5.12}$$

$$M_{el}^{(1)}(K^+ \to K^0\pi^+\pi^0) = -\frac{i}{2\sqrt{2}f_K^2} C_K + (m_K^2 - m_\pi^2)\left(\frac{1+\delta^2}{8}\ln\Lambda^2 + d_\pi\right) \tag{5.13}$$

where d_π is some finite term (keeping only linear terms in the parameter 'a' in the extrapolation). Because of the relation (5.6), the sum of the contributions (5.12) and (5.13) to the $K^+ \to \pi^+\pi^0$ decay amplitude is also finite.

To calculate the tadpole contributions to $M^{(2)}(K^+ \to K^0\pi^+\pi^0)$, we assume that the tadpole behaves like an u_3 term of the $(\bar{3},3)+(3,\bar{3})$ representation. Then

$$M_t^{(2)}(K^+ \to K^0\pi^+\pi^0) = \lim_{q,q'\to 0} M_t^{(2)}(K^+ \to K^0\pi^+\pi^0) = \frac{i}{2\sqrt{2}f_\pi^2}(\Delta m_K^2)_t. \tag{5.14}$$

In writing the first equality of (5.14) we have assumed that the coefficient of the u_3 term is independent of the parameter 'a', although in some models they can be related (Subba-Rao 1972).

6. Results and discussions

We are now in a position to write down the $\Delta I = \frac{3}{2}$ and $\frac{5}{2}$ contributions due to both the virtual electromagnetic and the weak interactions to the $K \to 2\pi$ decay amplitudes.

Remembering that the contributions from the graph of figure 1 have appropriate kaon propagators at zero momentum we have for A^{+}_{+0} from (5.12)–(5.14) and (5.9)

$$|A^{+}_{+0}| = \frac{2|a_0|}{m_{\mathrm{K}}^2}\{3(m_{\pi^+}^2 - m_{\pi^0}^2) - (m_{\mathrm{K}^+}^2 - m_{\mathrm{K}^0}^2)\} + \frac{Gf_\pi}{\sqrt{2}}(1+\delta^2)(m_{\mathrm{K}}^2 - m_\pi^2) \tag{6.1}$$

where we have used the fact that $C_\pi \simeq (m_{\pi^+}^2 - m_{\pi^0}^2)$ and the relation (Wallace 1969)

$$\lim_{q,q'\to 0} M_{\mathrm{W}}(\mathrm{K}^0 \to \pi^0\pi^0) = -\frac{\langle 0|H_{\mathrm{W}}|\mathrm{K}^0\rangle}{4f_\pi^2}\left(1 - \frac{\mathrm{O}(m_\pi^2)}{m_{\mathrm{K}}^2}\right).$$

We now use $f_\pi = 0{\cdot}67 m_\pi$, $G = 1{\cdot}026\times 10^{-5} m_{\mathrm{p}}^{-2}$ and $\delta = 0$ (favoured by Schwinger 1967) and $\sin\theta = 0{\cdot}26$ (to be consistent with the relation (5.6)) in (6.1) and express the results in terms of the parameters $|\omega|$, and $|\xi|$. $|(\delta_0-\delta_2)|$ is then determined from the relation (2.10). We get

$$|\omega| = 0{\cdot}040 \pm 0{\cdot}001 \tag{6.2}$$

$$|\xi| = 1{\cdot}17 \pm 0{\cdot}003 \tag{6.3}$$

$$|(\delta_0-\delta_2)| = 60° \pm 5° \tag{6.4}$$

which should be compared with the experimental values

$$|\omega|_{\mathrm{exp}} = 0{\cdot}035^{+0{\cdot}012}_{-0{\cdot}009} \tag{6.5}$$

$$|\xi|_{\mathrm{exp}} = 1{\cdot}35^{+0{\cdot}44}_{-0{\cdot}37} \tag{6.6}$$

$$(\delta_0-\delta_2)_{\mathrm{exp}} = 55{\cdot}5° \pm 5{\cdot}1°. \tag{6.7}$$

The value of $(\delta_0-\delta_2)_{\mathrm{exp}}$ in (6.7) is taken from the analysis of Marakov *et al* (1972). The values of $|\omega|_{\mathrm{exp}}$ and $|\xi|_{\mathrm{exp}}$ are then calculated from the graphs of figure 2. The predicted values of $|\omega|$, $|\xi|$ and $|(\delta_0-\delta_2)|$ are in good agreement with their experimental values (6.5)–(6.7), although the latter have still large errors‡.

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‡ A recent measurement of the K_S^0 lifetime by Skjeggestad *et al* (*Nucl. Phys.* B to be published) shows that it is larger by 3·9% compared to the world average one given in the reference (Söding *et al* 1972), used in equation (2.13) to calculate the values of b^+. The larger figure of Skjeggestad *et al* would therefore increase the values of $|\omega|$ in equations (6.2) and (6.5) by 1·9%, and $|(\delta_0-\delta_2)|$ by 0·6°, in equation (6.4).

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